ON THE SUBSPACES OF $L^p (p > 2)$ SPANNED BY SEQUENCES OF INDEPENDENT RANDOM VARIABLES

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ABSTRACT

Let $2 < p < \infty$. The Banach space spanned by a sequence of independent random variables in L^p, each of mean zero, is shown to be isomorphic to l^2, l^p, $l^2 \oplus l^p$, or a new space X_p, and the linear topological properties of X_p are investigated. It is proved that X_p is isomorphic to a complemented subspace of L^p and another uncomplemented subspace of L^p, whence there exists an uncomplemented subspace of l^p isomorphic to l^p. It is also proved that X_p is not isomorphic to the previously known $\mathscr{L}_p$ spaces.

1. Introduction

Fix $1 < p < \infty$, $p \neq 2$. The motivation for this work derives from the following question: What are the Banach spaces isomorphic (linearly homeomorphic) to a complemented subspace of L^p?

This question acquired added interest from the results of [7] and [10], which show that a Banach space has this property if and only if it is an $\mathscr{L}_p$ space or an $\mathscr{L}_2$ space (see the next section for the appropriate definitions). The previously known separable $\mathscr{L}_p$ spaces are L^p, l^p, $l^p \oplus l^2$, and $(l^2 \oplus l^2 \oplus \cdots)_p$. We construct here an $\mathscr{L}_p$ space, denoted X_p, which is considerably different from these four previously known spaces. (Thus problem 1d of [7] is answered in the negative). We discovered this space by investigating the span of a sequence of independent random variables in L^p of mean-zero, for $p > 2$. (X_p is defined following Theorem 3 of §3). We completely determine the norm-structure of the span of such a sequence in §3. The inequalities derived there (in Lemmas 1 and 2, and in Theorem 3), may be of independent interest to probability theorists.

Now fix $2 < p < \infty$. We also prove in §3 that X_p is isomorphic to a complemented subspace of L^p(namely the closed linear span of a certain sequence of

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independent 3-valued symmetric random variables), and an uncomplemented subspace $\tilde{X}_p$ of $l^p \oplus l^2$. In fact we prove that $(l^p \oplus l^2)/\tilde{X}_p$ is not an $\mathscr{L}_p$ (or an $\mathscr{L}_2$) space, thus yielding a partial answer to problem 4a of [10]. We also show that there exists an uncomplemented subspace of l^p isomorphic to l^p. (This shows in the terminology of [9], that l^p is not "subspace homogeneous" for these values of p). In the appendix we give a constructive proof of this result and the fact that $\tilde{X}_p$ is uncomplemented in $l^p \oplus l^2$, by means of studying matrices representing operators from l^p_n to l^2_n.

In §4 we study the linear topological properties of X_p. In particular we show that X_p is isomorphic to a "symmetric" sum of itself (thus X_p is isomorphic to its square, i.e. to $X_p \oplus X_p$), and we prove that X_p is not isomorphic to the four previously known $\mathscr{L}_p$ spaces mentioned above. This proof is rather delicate, and consists mainly in showing that X_p is not a continuous linear image of $(l^2 \oplus l^2 \oplus \cdots)_p$. Our argument for this requires the introduction of a new isomorphism invariant. This property (defined prior to Theorem 9) may be of use in other isomorphism problems. (See also a related concept defined in the second remark following Lemma 10.)

Our techniques show the existence of at least one more new $\mathscr{L}_p$ space, denoted B_p (defined at the end of §4). Thus we now have at least six mutually non-isomorphic separable $\mathscr{L}_p$-spaces (of infinite dimension). It seems quite possible that there are infinitely many isomorphically distinct separable $\mathscr{L}_p$ spaces of infinite dimension; this is of course an open problem.

Results of ours and others discovered since the writing of the first draft of the paper, show that the $\mathscr{L}_p$ space $(X_p \oplus X_p \oplus \cdots)_p$ is isomorphically distinct from these six spaces; see the end of §4 for further remarks.

2. Definitions and notation.

By "operator" we mean "bounded linear operator" and by "projection", "bounded linear projection". Given Banach spaces X and Y (over either the real or complex scalars), we say that the operator $T: X \to Y$ is an isomorphism (resp. an isometry) if T is 1-1 with closed range (resp. T is norm-preserving). We say that X and Y are isomorphic (denoted $X \sim Y$) if there exists an isomorphism from X onto Y; then we define $d(X, Y) = \inf\{\|T\| \, \|T^{-1}\| : T: X \to Y$ is a surjective isomorphism$\}$.

Given A a closed subspace of X, $A^\perp$ denotes the annihilator of A in X^*, the dual of X. (Thus $A^\perp = \{x^* \in X^* : x^*(a) = 0$ for all $a \in A\}$).

If (x_n) is a sequence of elements in X, $[x_n]$ denotes the closed linear span of the x_n's; $[x_1, \cdots, x_n]$ denotes the linear span of the n elements $x_1, \cdots, x_n$. (x_n) is said to be a basic sequence (resp. an unconditional basic sequence) if for every element $y \in [x_n]$ there exists a unique sequence of scalars (c_n) with $y = \sum c_n x_n$, the series converging in norm (resp. the series converging unconditionally in norm). (x_n) is said to be a basis of X if (x_n) is a basic sequence and $[x_n] = X$. For properties of bases and unconditional convergence, see [2].

Given Banach spaces X and Y with bases (x_n) and (y_n) respectively and $K < \infty$, we say that (x_n) and (y_n) are K-equivalent (resp. isometrically equivalent) if there exists an isomorphism (resp. an isometry) $T: X \to Y$ with $T(x_n) = y_n$ for all n, such that $\| T^{-1} \| \, \| T \| < K$. We say that (x_n) and (y_n) are equivalent if they are K-equivalent for some K. We note that by the closed graph theorem, (x_n) and (y_n) are equivalent if and only if for all sequences of scalars (c_n), $\sum c_n x_n$ converges if and only if $\sum c_n y_n$ converges.

Given $1 < p < \infty$ and Banach spaces $X_1, X_2, \cdots$, $(X_1 \oplus X_2 \oplus \cdots)_p$ denotes the Banach space consisting of all sequences $(x_n) \in \prod_{i=1}^{\infty} X_i$ with $\| (x_n) \| = (\sum_{j=1}^{\infty} \| x_j \|^p) < \infty$. (Elements of $\prod_{i=1}^{\infty} X_i$ may be denoted $(x(n))$ on occasion). $(X_1 \oplus \cdots \oplus X_n)_p$ is defined in the obvious way; if X_i is the one-dimensional space of scalars for all i, then $(X_1 \oplus \cdots \oplus X_n)_p$ is denoted by l_n^p and $(X_1 \oplus X_2 \oplus \cdots)_p$ by l^p. The *usual basis* or *unit-vectors basis of l^p* refers to the sequence (y_n) where $y_n(j) = \delta_{nj}$ for all positive integers n and j.

For $1 \leqq p < \infty$, L^p refers to $L^p[0,1]$; i.e. the space of equivalence classes of pth power Lebesgue integrable functions f on $[0, 1]$ under the norm $\| f \|_p = (\int_0^1 | f(t) |^p dt)^{1/p}$ *Throughout, p and q shall always denote numbers such that $1/p + 1/q = 1$.* We identify $(L^p)^*$ with L^q.

By a random-variable in L^p, we mean simply a measurable function f belonging to L^p. For the definition and standard results concerning independent random variables, see [11].

A Banach space X is called an $\mathscr{L}_p$ space if there exists a $\lambda > 1$ such that for every finite-dimensional subspace F of X, there exists an n and an n-dimensional subspace G of X with $F \subset G$ and $d(G, l_n^p) \leqq \lambda$. For the basic properties of $\mathscr{L}_p$-spaces, see [7] and [10]. We do not use these results in an essential way here. However, the basic result of [10] mentioned in the introduction motivates our work.

3. Inequalities concerning independent random variables in L^p

As we remarked in the introduction, it is known that l^2 is isomorphic to a complemented subspace of L^p for all $1 < p < \infty$. This may be seen explicitly as follows: Let $r(t)$ be the function with period one defined on the real line by

$$r(t) = 1, \quad 0 \leqq t < \tfrac{1}{2}$$

$$r(t) = -1, \quad \tfrac{1}{2} \leqq t < 1.$$

Let $r_n(t) = r(2^{n-1}t)$ for $n = 1, 2, \cdots$. The functions r_n are called the *Rademacher functions.* In 1923, Khintchine proved that they satisfy the following inequalities [5]:

For any $0 < p < \infty$, there exists a constant B_p depending only on p, so that for any n and n-scalars $c_1, \cdots, c_n$,

$$(\Delta) \quad B_p^{-1}\left(\int_0^1 \left|\Sigma\, c_i r_i(t)\right|^p dt\right)^{1/p} \leqq \left(\Sigma |c_i|^2\right)^{1/2} \leqq B_p \left(\int_0^1 \left|\Sigma\, c_i r_i(t)\right|^p dt\right)^{1/p}.$$

It follows immediately from these inequalities that if Y denotes the closed linear span of the r_n's in L_p, and $1 \leqq p < \infty$, then Y is isomorphic to l^2. Moreover, if $1 < p < \infty$, the orthogonal projection extends naturally to a bounded linear projection from L^p onto Y.

In the language of probability theory, $r_1, r_2, \cdots$ is a sequence of independent symmetric 2-valued random variables. We shall prove that if $f_1, f_2, \cdots$ is a sequence of independent symmetric 3-valued random variables and Y denotes its closed linear span in L_p, $(1 < p < \infty)$, then again Y is complemented by means of orthogonal projection. We shall also see that if $p > 2$, then putting $w_n = \|f_n\|_p / \|f_n\|_2$, if the sequence (w_n) satisfies

$$(1) \qquad \Sigma\, w_n^{2p/(p-2)} = \infty \quad \text{and} \quad w_n \to 0 \quad \text{as} \quad n \to \infty,$$

then Y is isomorphic to an uncomplemented subspace of $l^p \oplus l^2$. In section 4 we shall see that the isomorphism type of Y is independent of the sequence w satisfying (1) (in fact $Y \sim X_p$ of the introduction), and is considerably different from other known $\mathscr{L}_p$ spaces (e.g. Y is not a continuous linear image of $l^p \oplus l^2$).

We begin our work by investigating the Banach space structure of the span in L^p of a sequence (f_n) of independent random variables of mean zero. Our first three results show that this structure depends solely on the sequence of ratios (w_n) defined above.

LEMMA 1. *Let $1 < p < \infty$, and let $f_1, \cdots, f_n$ be independent random variables belonging to L^p. Then*

$$\left(\left| \int |f_1 + \cdots + f_n|^p dx \right)^{1/p} \leqq 2^p \max\left\{ \left(\int |f_1|^p dx + \cdots + \int |f_n|^p dx \right)^{1/p}, \right.$$

$$\left. \int |f_1|\, dx + \cdots + \int |f_n|\, dx \right\}.$$

PROOF. We introduce the following notation; for $0 \leqq r \leqq p$, put $\mu_r = (\int |f_1 + \cdots + f_n|^r dx)^{1/r}$ and

$$N_p = \left(\int (|f_1|^p + \cdots + |f_n|^p) dx \right)^{1/p}.$$

We may and shall assume that $f_i \geqq 0$ for all i. Now

$$\mu_p^p = \int (f_1 + \cdots + f_n)^{p-1}(f_1 + \cdots + f_n) dx.$$

But

$$\begin{aligned} \int (f_1 + \cdots + f_n)^{p-1} f_1 dx &\leqq 2^{p-1} \int [f_1^{p-1} + (f_2 + \cdots + f_n)^{p-1}] f_1 dx \\ &= 2^{p-1} \left[\int f_1^p dx + \int (f_2 + \cdots + f_n)^{p-1} dx \int f_1 dx \right] \\ &\leqq 2^{p-1} \left[\int f_1^p dx + \int (f_1 + \cdots + f_n)^{p-1} dx \int f_1 dx \right], \end{aligned}$$

the equality holding by the independence of f_1 and $(f_2 + \cdots + f_n)$. Similarly,

$$\int (f_1 + \cdots + f_n)^{p-1} f_i dx \leqq 2^{p-1} \left[\int f_i^p dx + \mu_{p-1}^{p-1} \int f_i dx \right]$$

holds for any i, $1 \leqq i \leqq n$. Summing over i, we obtain that

$\mu_p^p \leqq 2^{p-1}(N_p^p + \mu_{p-1}^{p-1} N_1)$. But $\mu_{p-1} \leqq \mu_p$ by Hölders inequality. Thus

$$\mu_p^p \leqq 2^{p-1}(N_p^p + \mu_p^{p-1} N_1) \leqq 2^p \max\{N_p^p, \mu_p^{p-1} N_1\}$$

or

$$\mu_p \leqq \max\{2N_p, 2^p N_1\} \leqq 2^p \max\{N_p, N_1\}. \qquad \text{Q.E.D.}$$

REMARK. For fixed p and non-negative f_i's the inequality of Lemma 1 is sharp to within a constant. For the non-negativity of the f_i's alone, implies that $\mu_p \geqq \max\{N_p, N_1\}$.

The next lemma is essentially known, i.e. it is a simple consequence of known results.

LEMMA 2. *Let $1 \leqq p < \infty$ and let $f_1, \cdots, f_n$ be independent random variables belonging to L^p with $\int f_i dx = 0$ for all i.*

(a) *If $\varepsilon_1, \cdots, \varepsilon_n$ are given with $\varepsilon_i = \pm 1$ all i, then*

$$\left(\int |\varepsilon_1 f_1 + \cdots + \varepsilon_n f_n|^p dx\right)^{1/p} \leqq 2\left(\int |f_1 + \cdots + f_n|^p dx\right)^{1/p}$$

(b) $$\left(\int |f_1 + \cdots + f_n|^p dx\right)^{1/p} \leqq 2\left(\int |f_1|^p dx + \cdots + \int |f_n|^p dx\right)^{1/p} \text{ for } p < 2$$

$$\left(\int |f_1 + \cdots + f_n|^p dx\right)^{1/p} \geqq \tfrac{1}{2}\left(\int |f_1|^p dx + \cdots + \int |f_n|^p dx\right)^{1/p} \text{ for } p > 2.$$

PROOF. (a) By a standard result in probability theory, if f and g are independent random variables in L^p with $\int f = \int g = 0$, then $\int |f|^p dx \leqq \int |f + g|^p dx$ (c.f. p. 263 of [11]). Thus $\|f\| \leqq \|f + g\|$. ($\|\cdot\|$ denotes here, the p-norm of f). Now let $\varepsilon_1, \cdots, \varepsilon_n$ be given, and put $f = \sum_{\varepsilon_i = +1} f_i$ and $g = \sum_{\varepsilon_i = -1} f_i$. Then $f + g = \sum_{i=1}^n f_i$ and $f - g = \sum_{i=1}^n \varepsilon_i f_i$. Our assumptions imply that $\int f = \int g = 0$ and f and g are independent, whence $\|f\| \leqq \|f + g\|$ and $\|g\| \leqq \|f + g\|$, so $\|f - g\| \leqq 2\|f + g\|$.

(b) This is an immediate consequence of (a) and the following known result: If $g_1, \cdots, g_n \in L^p$ are given, then if $p < 2$, there exist $\varepsilon_i = \pm 1$ with

$$\|\varepsilon_1 g_1 + \cdots + \varepsilon_n g_n\|_p \leqq (\|g_1\|_p^p + \cdots + \|g_n\|_p^p)^{1/p},$$

while if $p > 2$ there exist $\varepsilon_i = \pm 1$ with
$(\|g_1\|_p^p + \cdots + \|g_n\|_p^p)^{1/p} \leqq \|\varepsilon_1 g_1 + \cdots + \varepsilon_n g_n\|_p$. These are both in turn simple consequences of Clarkson's inequalities. For $p < 2$, an explicit proof may be found on p. 209 of [16]. For $p > 2$, an argument almost identical to the one in [16] may be obtained by simply reversing all the inequalities given there, starting with Clarkson's inequality valid for $p > 2$;

$$\|f\|_p^p + \|g\|_p^p \leqq \tfrac{1}{2}(\|f + g\|_p^p + \|f - g\|_p^p).$$

REMARK 1. If the f_i's are in addition assumed to be symmetric, then in Lemma 2, the constant 2 may be replaced by the constant 1.

REMARK 2. Lemma (2a) implies immediately that if $f_1, f_2, \cdots$ is an infinite sequence of independent random variables in L^p, each of mean zero, then (f_n) is an unconditional basic sequence.

THEOREM 3. *Let $2 < p < \infty$. Then there exists a constant K_p depending* *only on p, so that if $f_1,\cdots,f_n$ are independent random variables belonging to L^p with $\int f_i dx = 0$ for all i, then*

$$\left(\int \left| \sum_{i=1}^{n} f_i \right|^p dx\right)^{1/p} \leqq K_p \max\left\{\left(\sum_{i=1}^{n} \int |f_i|^p dx\right)^{1/p}, \left(\sum_{i=1}^{n} \int |f_i|^2 dx\right)^{1/2}\right\}$$

and

$$\left(\int \left| \sum_{i=1}^{n} f_i \right|^p dx\right)^{1/p} \geqq \tfrac{1}{2} \max \left\{\left(\sum_{i=1}^{n} \int |f_i|^p dx\right)^{1/p}, \left(\sum_{i=1}^{n} \int |f_i|^2 dx\right)^{1/2}\right\}.$$

PROOF. We adopt the same notation employed in the proof of Lemma 1. Thus, we are to prove that $\mu_p \leqq K_p \max\{N_p, N_2\}$ for some K_p depending only on p, and $\mu_p \geqq \frac{1}{2}\max\{N_p, N_2\}$. Now the independence of the f_i's and our assumptions imply that $\int f_i f_j dx = \int f_i dx \int f_j dx = 0$ all $i \neq j$, i.e. the f_i's are orthogonal. Thus $N_2 = \mu_2$ and hence the second inequality follows from Lemma 2(b) and the fact that $\mu_2 \leqq \mu_p$.

To prove the first inequality, we let $r_1,\cdots,r_n$ be the first n Rademacher functions (as defined at the beginning of this section), and fix $0 \leqq t \leqq 1$. Then by Lemma 2(a),

$$\mu_p^p \leqq 2^p \int \left| r_1(t)f_1(x) + \cdots + r_n(t)f_n(x)\right|^p dx.$$

Integrating both sides of this inequality with respect to t and changing the order of integration, we obtain

$$\mu_p^p \leqq 2^p B_p^p \int (|f_1|^2(x) + \cdots + |f_n|^2(x))^{p/2} dx$$

by Khintchine's inequalities (Δ) (where B_p is a constant depending only on p, in fact $B_p \leqq \sqrt{p/2)}$. But $|f_1|^2,\cdots,|f_n|^2$ are independent non-negative random variables belonging to $L^{p/2}$. Since $p/2 > 1$, we obtain by Lemma 1 that

$$\left(\int (|f_1|^2 + \cdots + |f_n|^2)^{p/2} dx\right)^{2/p} \leqq 2^{p/2} \max\{N_p^2, N_2^2\}.$$

Thus the last two inequalities yield that $\mu_p \leqq 2\, B_p 2^{p/4}\{\max N_p, N_2\}$, so we may set $K_p = 2^{p/4+1} B_p$. Q.E.D.

(The technique of integrating against the Rademacher functions is well known; it may be found in Paley's work [12].)

Let $2 < p < \infty$. Theorem 3 determines the structure of the Banach space spanned in L^p by a sequence of independent random variables of mean-zero.

Precisely, let $f_1, f_2, \cdots$ be such a sequence, put $w_i = \| f_i \|_2 / \| f_i \|_p$ and assume (as we may) that $\| f_i \|_p = 1$ for all i. Letting Y denote the closed linear span of the f_i's in L^p, Theorem 3 shows that if $y \in Y$, there exists a unique sequence of scalars (x_n) such that $y = \sum_{i=1}^{\infty} x_i f_i$, the series converging in L^p mean (and incidentally a.e. by the independence of the f_i's), such that

$$\sum |x_i|^p < \infty \text{ and } \sum |x_i|^2 w_i^2 < \infty. \tag{2}$$

Moreover, defining $\| (x_n) \|$ by

$$\| (x_n) \| = \max\{ (\sum |x_i|^p)^{1/p}, (\sum |x_i|^2 w_i^2)^{1/2} \},$$

we have that

$$\tfrac{1}{2} \| (x_n) \| \leqq \| y \|_p \leqq K_p \| (x_n) \|.$$

Accordingly, given any sequence $w = (w_n)$ of positive scalars, *we define $X_{p,w}$ to be the space of all sequences (x_n) of scalars satisfying* (2), *under the norm defined above.* It is easily verified directly that $X_{p,w}$ is a Banach space; we shall see shortly that if w satisfies(1), $X_{p,w}$ is isometric to a closed uncomplemented subspace of $l^p \oplus l^2$ (where we define $\| x + y \| = \max\{ \| x \|, \| y \| \}$ for $x \in l^p$, $y \in l^2$). Moreover we shall show in §4 that $X_{p,w} \sim X_{p,w'}$ if w, w' satisfy (1). Thus the space X_p mentioned in the introduction refers to $X_{p,w}$ for any w satisfying (1); X_q is defined by duality, $X_q = X_p^*$.

The point of our development so far, is that if the f_i's, w_i's, and Y are as above, then Y is isomorphic to $X_{p,w}$.

We wish now to show that for any w, $X_{p,w}$ is isomorphic to a complemented subspace of L^p. Towards this end, we need to study the spaces spanned by symmetric 3-valued independent random variables.

We recall that a random variable g defined on $[0,1]$ is said to be symmetric if $\mu\{x : g(x) \in E\} = \mu\{x : -g(x) \in E\}$ for any Lebesgue measurable set E, where μ denotes Lebesgue measure. (Evidently if g is symmetric and integrable, then $\int g dx = 0$). Now suppose that g is a symmetric random variable taking on the values 1, 0 and -1. Then $|g|$ is the characteristic function of some measurable subset of $[0,1]$; thus if $2 < p < \infty$,

$$\| \| g \|_2^{-2} g \|_q = \| g \|_p^{-1} \left(\text{where as always, } \frac{1}{p} + \frac{1}{q} = 1 \right). \tag{3}$$

Evidently (3) also holds for any symmetric 3-valued random variable g.

We are now prepared to prove that $X_{p,w}$ is indeed isometric to a complemented subspace of L^p, $(p > 2)$.

THEOREM 4. *Let $1 < p < \infty$, let $f_1, f_2, \cdots$ be an infinite sequence of independent symmetric 3-valued random variables, and let Y_p denote their closed linear span in L^p. Then there exists a projection P from L^p onto Y_p such that $\| P \| \leqq K_p$ (where K_p is as in Theorem 3). Moreover fixing $p > 2$ and putting $w_n = \| f_n \|_2 / \| f_n \|_p$ for all n, Y_p is isomorphic to $X_{p,w}$ while Y_q is isomorphic to $X^*_{p,w}$.*

PROOF. Fix $2 < p < \infty$. We shall prove that orthogonal projection yields a projection from L^p onto Y_p of norm at most K_p. We then easily obtain all the assertions of Theorem 4. For the adjoint of this projection yields a projection from L^q onto Y_q, and of course then $Y_q \sim Y_p^* \sim X^*_{p,w}$ by the remarks following Theorem 3.

We may assume without loss of generality that f_n is real-valued with $\| f_n \|_p = 1$ for all n. We then define P to be the restriction to L^p, of the orthogonal projection from L^2 onto Y_2, regarding $L^p \subset L^2$. Thus $P: L^p \to L^2$ is explicitly given by

$$P(f) = \sum_{n=1}^{\infty} \left(\int_0^1 f(x) f_n(x) dx \right) \| f_n \|_2^{-2} f_n$$

for all $f \in L^p$. Now fix $f \in L^p$, and put

$$x_n = \left(\int_0^1 f(x) f_n(x) dx \right) \| f_n \|_2^{-2} \quad \text{for all } n.$$

Then

$$(\Sigma \, |x_n|^2 w_n^2)^{1/2} = \| Pf \|_2 \leqq \| f \|_2 \leqq \| f \|_p .$$

Now let n be fixed and $c_1, \cdots, c_n$ be n scalars such that $\Sigma \, |c_j|^q \leqq 1$. Then putting $\tilde{f}_j = \| f_j \|_2^{-2} f_j$, $c_1 \tilde{f}_1, c_2 \tilde{f}_2, \cdots, c_n \tilde{f}_n$ are independent random variables belonging L^q, each of mean zero. Since $q < 2$ and $\| c_j \tilde{f}_j \|_q = |c_j|$ for all j by (3), Lemma 2(b) (or rather the first remark following Lemma 2) implies that

$$\| \, \Sigma \, c_j \tilde{f}_j \|_q \leqq \Sigma \, |c_j|^q \leqq 1 .$$

Thus

$$| \, \Sigma \, c_j x_j | = \left| \int_0^1 f(x) (\Sigma c_j \tilde{f}_j) dx \right| \leqq \| f \|_p .$$

Hence since n and $c_1, \cdots, c_n$ were arbitrary satisfying $\Sigma \, |c_j|^q \leqq 1$,

$$\left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} \leqq \| f \|_p .$$

Thus (x_j) belongs to $X_{p,w}$, and $\|(x_j)\| \leqq \|f\|_p$. Hence by Theorem 3 and our remarks following it, $Pf = \sum x_j f_j$ belongs to Y_p, and $\|Pf\|_p \leqq K_p \|f\|_p$. Thus P yields a projection from L^p onto Y of norm at most K_p. Q.E.D.

REMARK. Theorem 4 holds also for any sequence of independent complex-valued random variables f_n of mean zero such that $|f_n|$ is $\{0,1\}$-valued for all n (here we obtain $\|P\| \leqq 2K_p$). However we shall see shortly (in the corollary following Proposition 5) that in general, the span in L^p of a sequence of independent random variables of mean zero, need not be complemented.

COROLLARY. *Let $w = (w_n)$ be any sequence of positive scalars, and let $2 < p < \infty$, Then $X_{p,w}$ is isomorphic to a complemented subspace of L^p. If $\inf_n w_n = 0$, then $X_{p,w}$ is an $\mathscr{L}_p$ space.*

PROOF. First assume that $0 < w_n \leqq 1$ for all n. Rather than considering $L^p[0,1]$, consider L^p of the measure space $\prod_{n=1}^{\infty}\{-1,0,1\}$ with product measure $\prod_{n=1}^{\infty}\mu_n$ where for all n, μ_n is the measure on $\{-1,0,1\}$ defined by

$$\mu_n\{-1\} = \mu_n\{1\} = \tfrac{1}{2} w_n^{2p/(p-2)} \text{ and } \mu_n\{-1,0,1\} = 1.$$

Then put $f_n(x) = x_n$ for all $x \in \prod_{j=1}^{\infty}\{-1,0,1\}$. The sequence $f_1, f_2, \cdots$ satisfies the hypotheses of Theorem 4, and $w_n = \|f_n\|_2 / \|f_n\|_p$ for all n. (Of course, one can also explicitly represent the f_n's as being defined on $[0,1]$ itself).

To pass to the general case, we first observe that if $w_n > 1$ for all n, then $X_{p,w}$ is isometric to l^2. Consequently $X_{p,w}$ is isomorphic to a complemented sequence of L^p. Finally, if w is an arbitrary sequence, we let w', w'' be its subsequences such that $w'(n) \leqq 1$ for all n and $w''(n) > 1$ for all n. Then

$$X_{p,w} \sim X_{p,w'} \oplus X_{p,w''} \text{ and } X_{p,w'} \oplus X_{p,w''}$$

is isomorphic to a complemented subspace of $L^p \oplus L^p$ which is in turn isomorphic to L^p. The second assertion of the corollary is an immediate consequence of the above and Theorem III of [10]. Q.E.D.

REMARK: There exists a single sequence of simple functions whose span in the different L^s spaces is isomorphic to the space X_s mentioned in the introduction. For let (λ_n) be a sequence of numbers with $0 < \lambda_n < 1$, $\lambda_n \to 0$, and $\sum \lambda_n = \infty$. Let $f_1, f_2, \cdots$ be a sequence of independent symmetric $\{-1,0,1\}$-valued random variables with $\int_0^1 |f_n| dt = \lambda_n$ for all n. Let Y_s denote the closed linear span of the f_n's in L^s. Then fixing $2 < p < \infty$, Y_p (resp. Y_q) is complemented and isomorphic to $X_{p,w}$ (resp. to $X^*_{p,w}$) where $(w_n)=(\lambda_n^{(p-2)/(2p)})$, by Theorem 4. Thus $Y_s \sim X_s$ for all $1 < s < \infty$ (of course w satisfies (1) for fixed $2 < p < \infty$).

We pass now to the proof that $X_{p,w}$ is isomorphic to an uncomplemented subspace of L^p if w satisfies (1). To motivate the introduction of (1) we observe that if $\sum w_n^{2p/(p-2)} < \infty$ (for fixed $p > 2$), then by Hölder's inequality,

$$\sum |x_n|^2 w_n^2 \leqq (\sum |x_n|^p)^{2/p} (\sum w_n^{2p/(p-2)})^{(p-2)/p}$$

for any sequence of scalars (x_n), and consequently $X_{p,w}$ is isomorphic to l^p in this case. If $\inf_n w_n > 0$, then $X_{p,w}$ is obviously isomorphic to l^2, so consequently if the positive integers split into two disjoint infinite subsequences (n_i) and (m_i) such that $\sum w_{n_i}^{2p/(p-2)} < \infty$ and $\inf w_{m_i} > 0$, $X_{p,w}$ is isomorphic to $l^p \oplus l^2$, None of these possibilities occur if and only if w satisfies the conditions

$$(1') \qquad \varliminf_{n\to\infty} w_n = 0 \quad \text{and} \quad \sum_{w_n<\varepsilon} w_n^{2p/(p-2)} = \infty \quad \text{for all } \varepsilon > 0.$$

(1) is of course the simplest way in which (1′) can occur.

We shall prove later that if w, w' both satisfy (1′), then $X_{p,w} \sim X_{p,w'}$. For the present, we wish to show that if w satisfies (1), then $X_{p,w}$ is isomorphic to an uncomplemented subspace of $l^p \oplus l^2$.

Throughout the end of the next proposition, p is fixed with $2 < p < \infty$. Let (e_n) (resp. (b_n)) denote the unit-vectors-basis in l^p (resp. in l^2). Given $w = (w_n)$, for each n let $d_n = e_n + w_n b_n$, and let $\tilde{X}_{p,w}$ denote $[d_n]$ in $l^p \oplus l^2$ (we norm $l^p \oplus l^2$ by $\|(x,y)\| = \max(\|x\|, \|y\|)$ if $x \in l^p$ and $y \in l^2$.) It is immediate that $X_{p,w}$ and $\tilde{X}_{p,w}$ are isometric under the canonical map $(x_n) \to \sum_{n=1}^{\infty} x_n d_n$. (It is also easily seen that the spaces $X_{p,w}$ for arbitrary w, are isometric to all spaces in $l^p \oplus l^2$ spanned by a "block basis" of $e_1, b_1, e_2, b_2, \cdots$).)

PROPOSITION 5. *If w satisfies* (1), *then $(l^p \oplus l^2)/\tilde{X}_{p,w}$ is not isomorphic to a subspace of L^p. Consequently $\tilde{X}_{p,w}$ is uncomplemented in $l^p \oplus l^2$.*

PROOF. Let (e_n^*) (resp. (b_n^*)) be the functionals biorthogonal to the e_n's (resp. the b_n's). Thus (e_n^*) (resp. (b_n^*)) may be identified with the usual basis of l^q (resp. l^2).

For each n, put $a_n = w_n e_n^* - b_n^*$. Then (a_n) is a semi-normalized unconditional basis for $(\tilde{X}_{p,w})^\perp$ with the following property:

(4) (a_n) is not equivalent to (b_n), yet if (n') is any increasing sequence of indices, there exists an increasing subsequence (n'') of (n') with $(a_{n''(j)})$ equivalent to (b_j).

Indeed, for any sequence of scalars (x_n), $\Sigma x_n a_n$ converges if and only if $\Sigma |x_n|^q |w_n|^q < \infty$ and $\Sigma |x_n|^2 < \infty$; then $\| \Sigma x_n a_n \| = (\Sigma |x_n|^q |w_n|^q)^{1/q} + (\Sigma |x_n|^2)^{1/2}$. If (a_n) were equivalent to (b_n), there would exist a $K > 0$ such that $\| \Sigma x_n a_n \| \leqq K(\Sigma |x_n|^2)^{1/2}$ for any sequence of scalars (x_n). Then fixing N and putting $x_j = w_j^{q/(2-q)}$ for $1 \leqq j \leqq N$, we have that $\Sigma_{j=1}^N x_j^q w_j^q = \Sigma_{j=1}^N w_j^{2q/(2-q)} = \Sigma_{j=1}^N x_j^2$. Thus we would have that $(\Sigma_{j=1}^N w_j^{2q/(2-q)})^{1/q} \leqq K (\Sigma_{j=1}^N w_j^{2q/(2-q)})^{1/2}$ or $\Sigma_{j=1}^N w_j^{2q/(2-q)} \leqq K^{2q/(q-2)}$, whence $\Sigma_{j=1}^\infty w_j^{2q/(2-q)} < \infty$. But $2q/(2-q) = 2p/(p-2)$, thus our assumption that (w_n) satisfies (1) would be contradicted. Hence (a_n) is not equivalent to (b_n).

Now let (n') be an increasing sequence of indices. Since $w_n \to 0$, we may choose (n'') an increasing subsequence of (n') with $\Sigma_{j=1}^\infty w_{n''(j)}^{2q/(2-q)} < \infty$. Then by Hölder's inequality, $\Sigma |x_j|^q w_{n''(j)}^q \leqq (\Sigma |x_j|^2)^{q/2} (\Sigma w_{n''(j)}^{2q/(2-q)})^{(2-q)/2}$ for any scalars (x_j). Thus $\Sigma x_j a_{n''(j)}$ converges if and only if $\Sigma |x_j|^2 < \infty$; thus we have proved that (a_n) satisfies (4).

Now let (a_n^*) be the members of $[(\tilde{X}_{p,w})^\perp]^*$ biorthogonal to the a_n's. Then (a_n^*) is also a semi-normalized unconditional basic for $[(\tilde{X}_{p,w})^\perp]^*$, satisfying (4) (with "$a$" replaced by "$a^*$"). Now it follows from results of Kadec and Pelczynski (Theorems 2 and 3 of [4]), that if (z_n) is a semi-normalized unconditional basic sequence in L^p, then either (z_n) is equivalent to (b_n) or some subsequence of (z_n) is equivalent to (e_n). We have thus proved that $[(\tilde{X}_{p,w})^\perp]^*$ is isomorphic to no subspace of L^p, since (a_n^*) is equivalent to no unconditional basic sequence in L^p. Of course $[(\tilde{X}_{p,w})^\perp]^*$ is isometric to $(l^p \oplus l^2)/\tilde{X}_{p,w}$, so the proof is complete.

Remarks. 1. Assume w satisfies (1). Now it is known that l^2 is isometric to a subspace of L^p (cf. [7]). Thus if we norm $l^p \oplus l^2$ by $|||x \oplus y||| = (\| x \|^p + \| y \|^p)^{1/p}$ if $x \in l^p$ and $y \in l^2$, then $(\tilde{X}_{p,w}, |||\cdot|||)$ is isometric to an uncomplemented subspace of L^p. We suspect that $(\tilde{X}_{p,w}, |||\cdot|||)$ is not isometric to any complemented subspace of L^p.

2. We obtain from Proposition 5 a space with an unconditional basis, namely $l^p \oplus l^2$ and its basis $(e_1, b_1, e_2, b_2, \cdots)$, and an uncomplemented subspace spanned by a block basis with the blocks of length two, namely $X_{p,w}$ (for w satisfying (1)). See the second remark following Proposition 11 below, for further observations.

Corollary. *There exists a sequence $f_1, f_2, \cdots$ of independent 6-valued symmetric random variables, each of mean zero, such that Y_p, their closed linear span in L^p, is uncomplemented for all p, $2 < p < \infty$.*

PROOF. Let $g_1, g_2, \cdots$ be a sequence of independent symmetric random variables so that for all $n = 2,3,\cdots$, g_{2n-2} is $\{-1,1\}$ valued, g_{2n-3} is $\{-1,0,1\}$-valued, and $\int_0^1 |g_{2n-3}|\,dt = (n\log^2 n)^{-1}$. Let $f_{n-1} = g_{2n-3} + (1/\sqrt{n})\,g_{2n-2}$ for all such n.

Fixing $p > 2$, we have that $\| g_{2n-3}\|_2 / \| g_{2n-3}\|_p = (n\log^2 n)^{1/p-1/2}$ since $|g_{2n-3}|$ is the characteristic function of a measurable set, whence

$$\sum_{n=2}^{\infty} (\| g_{2n-3}\|_2/\| g_{2n-3}\|_p)^{2p/(p-2)} = \sum_{n=2}^{\infty} n^{-1}\log^{-2} n < \infty .$$

Let Z_p denote the closed linear span of the g_n's in L^p; then by Theorem 4 and the proof of the Corollary immediately following, $T\colon l^p \oplus l^2 \to Z_p$ is an isomorphism, where T is defined by

$$T\left(\sum_{n=1}^{\infty} (\alpha_n e_n + \beta_n b_n)\right)$$

$$= \sum_{n=1}^{\infty} (\alpha_n \| g_{2n-1}\|_p^{-1} g_{2n-1} + \beta_n g_{2n})$$

for all scalars α_n, β_n such that $\sum |\alpha_n|^p < \infty$ and $\sum |\beta_n|^2 < \infty$.

Moreover, letting $w = (w_n)$ be defined by $w_{n-1} = (\log n)^{2/p} n^{1/p-1/2}$ for all $n = 2,3,\cdots$, then w satisfies (1) and $T(\tilde{X}_{p,w}) = Y_p$. Since $\tilde{X}_{p,w}$ is uncomplemented in $l^p \oplus l^2$ by Proposition 5, Y_p is uncomplemented in Z_p (and consequently in L^p). Q.E.D.

REMARK. By the proof of Theorem 4, Z_q is complemented in L^q and "naturally" isomorphic to Z_p^* i.e. to $l^q \oplus l^2$. It is then easily seen that for all $1 < q \leqq 2$, Y_q is complemented in L^q and isomorphic to l^q. Now if we define $f'_{n-1} = g_{2n-3} + 1/(\sqrt{n}\log^2 n)\,g_{2n-2}$ for all $n \geqq 2$ and let Y'_r be the closed linear span of the f'_n's in L^r, then again Y'_p is complemented in L^p and isomorphic to l^p for all $2 \leqq p < \infty$. Moreover fixing $1 < q < 2$, Y'_q is isomorphic to no complemented subspace of L^q, by Proposition 5. For it can be seen that Y'_q is isomorphic to $(\tilde{X}_{p,w})^{\perp}$, where w is as in the proof of the Corollary.

We have now demonstrated that for all $2 < p < \infty$, there exists a complemented subspace of L^p isomorphic to an uncomplemented subspace of L^p. This result alone is sufficient to deduce the analogous fact for l^p. Because of our knowledge of $X_{p,w}$ given by Proposition 5, we can obtain more information, yielding a partial answer to problem 4a of [10].

THEOREM 6. *Let $2 < p < \infty$. Then there exists a closed subspace X of l^p*

such that X is isomorphic to l^p, yet l^p/X is isomorphic to no subspace of L^p. (Consequently X is uncomplemented in l^p).

PROOF. Let (w_n) satisfy (1), let $Z_n = \operatorname{span}\{e_j, b_j\colon 1 \leqq j \leqq n\}$ and $X_n = \operatorname{span}\{e_j + w_j b_j\colon 1 \leqq j \leqq n\}$ in $l^p \oplus l^2$. (Thus Z_n equals $l^p_n \oplus l^2_n$ and X_n equals the span of the first n-basis elements of $X_{p,w}$). We shall prove that

(a) $$\left(\sum_{n=1}^{\infty} \oplus Z_n\right)_p \Big/ \left(\sum_{n=1}^{\infty} \oplus X_n\right)_p$$

is isomorphic to no subspace of L^p;

(b) $$\left(\sum_{n=1}^{\infty} \oplus Z_n\right)_p \text{ and } \left(\sum_{n=1}^{\infty} \oplus X_n\right)_p$$

are each isomorphic to l^p. Theorem 6 follows immediately from (a) and (b).

To see (a), put $d_n = \inf\{d(Z_n/X_n, Y)\colon Y \text{ is an } n\text{-dimensional subspace of } L^p\}$. We shall show that $d_n \to \infty$. Letting (a_n^*) be as in the proof of Proposition 5, the span of $a_1^*, \cdots, a_n^*$ in $[(\tilde{X}_{p,w})^\perp]^*$ is isometric to Z_n/X_n. Evidently $d_n \leqq d_{n+1}$ for all n. If there were a $\lambda < \infty$ with $d_n \leqq \lambda$ for all n, we would obtain that for every finite dimensional $F \subset [(\tilde{X}_{p,w})^\perp]^*$, there would exist a $B \subset L^p$ with $d(F,B) \leqq 2\lambda$. Then by a result of Lindenstrauss and Pelczynski (Corollary 2, p. 306 of [7]), we would obtain that $[(\tilde{X}_{p,w})^\perp]^*$ is isomorphic to a subspace of L^p, contradicting Proposition 5. Thus $d_n \to \infty$, and (a) is proved.

(b) is an easy consequence of Theorem 4 and a result of Pelczynski. His result implies the following

LEMMA. *Let $1 \leqq p < \infty$, and $Y_1, Y_2, \cdots$ be a sequence of non-zero finite dimensional Banach spaces, such that there is a constant K, and for all n a subspace W_n of L^p, with $d(Y_n, W_n) \leqq K$ and a projection $P_n\colon L^p \to W_n$ with $\| P_n \| \leqq K$. Then $(\sum_{n=1}^{\infty} \oplus Y_n)_p$ is isomorphic to l^p.*

The lemma implies (b), for by Theorem 4 setting $Y_n = Z_n$ or $Y_n = X_n$ for all n, and letting $2 < p < \infty$, then $Y_1, Y_2, \cdots$ satisfies its hypotheses.

To see the lemma, we obviously have that $(\sum_{n=1}^{\infty} \oplus Y_n)_p$ is isomorphic to $(\sum_{n=1}^{\infty} \oplus W_n)_p$. For each n, we may choose a subspace F_n of L^p, such that letting $m_n = \dim F_n$, $d(F_n, l^p_{m_n}) \leqq 2$ and $W_n \subset F_n$. Thus $(\sum_{n=1}^{\infty} \oplus W_n)_p$ is complemented in $(\sum_{n=1}^{\infty} \oplus F_n)_p$ and obviously $(\sum_{n=1}^{\infty} \oplus F_n)_p$ is isomorphic to l^p. Hence $(\sum_{n=1}^{\infty} \oplus Y_n)_p$ is isomorphic to an infinite-dimensional complemented subspace of l^p, so $(\sum_{n=1}^{\infty} \oplus Y_n)_p$ is isomorphic to l^p by Theorem 1 of [14]. Q.E.D.

By the results of [7], every infinite-dimemsional $\mathscr{L}_p$-space contains a complemented subspace isomorphic to l^p, $1 \leqq p < \infty$. It can also be deduced by a compactness argument that if l^p contains an uncomplemented subspace isomorphic to l^p, then there exists a constant λ_p such that for all M, there exist integers $m < n$ and a subspace B of l^p_n with $d(B, l^p_m) \leqq \lambda_p$, such that if P is a projection from l^p_n onto B, then $\|P\| \geqq M$. Now it was previously known that l^p contains an uncomplemented subspace isomorphic to l^p if $1 < p < \frac{4}{3}$ (c.f. [9]; indeed this is a consequence of the known fact that for $1 < p < \frac{4}{3}$, L^p contains an uncomplemented subspace isomorphic to l^2, c.f. p. 52 of [18]). The following corollary may be deduced from the above facts and Theorem 6.

COROLLARY. *Let $1 < p < \frac{4}{3}$ or $2 < p < \infty$ and let B be an infinite-dimensional $\mathscr{L}_p$ space. Then there exists an uncomplemented subspace of B, isomorphic to l^p. If B is separable, there exists a sequence $B_1, B_2, \cdots$ of finite dimensional subspaces of B, with $B_1 \subset B_2 \subset \cdots$ and $B = \cup B_n$ and a constant $\lambda > 0$, such that $d(B_n, l^p_{m_n}) \leqq \lambda$ where $m_n = \dim B_n$; and such that $\rho_n \to \infty$, where $\rho_n = \inf\{\|P\| : P$ is a projection of B onto $B_n\}$.*

REMARKS. 1. The isometric version of the above corollary applied to $L^p(v)$ spaces, fails to be true. Indeed by using the appropriate analogue of Lemma 3 of [14] for $L^p(\mu)$ and a compactness argument involving the weak* operator topology (as in [6]), one can see the following: *Let μ and v be measures on possibly different measurable spaces, $1 \leqq p < \infty$, and A be a closed subspace of $L^p(v)$ isometric to $L^p(\mu)$. Then there is a projection from $L^p(v)$ onto A of norm one.* (For the case $p = 1$, one must also use that $L^1(\mu)$ is the range of a contractive projection in its double dual). (The fact that isometric imbeddings of l^p in L^p are complemented, is due to Pelczynski [14]).

2. In the following remarks, let $p > 2$, and assume w satisfies (1). In the appendix, we give a direct constructive proof of the facts that $\tilde{X}_{p,w}$ is uncomplemented in $l^p \oplus l^2$, and that $(\sum_{n=1}^{\infty} \oplus X_n)_p$ is uncomplemented in $(\sum_{m=1}^{\infty} \oplus Z_n)_p$. We shall also prove later, independently of the above reasoning, that $X_{p,w}$ is not a continuous linear image of $l^p \oplus l^2$. (Thus three independent proofs are given that $X_{p,w}$ is uncomplemented in $l^p \oplus l^2$).

3. It follows from Proposition 5 that $\tilde{X}^{\perp}_{p,w}$ is isomorphic to no complemented subspace of L^q, and from the proof of Theorem 6 that $(\sum_{n=1}^{\infty} \oplus [a_1, \cdots a_n])_q$ is not isomorphic to l^q (where $a_1, a_2, \cdots$ is as defined in the proof of Proposi-

tion 5). This seems to yield the most easily proved example of the existence (previously known) of a closed subspace of l^q of infinite dimension, non-isomorphic to l^q, for all $1 < q < 2$. (The existence of such a subspace for $2 < q < \infty$ remains an open problem. Such a subspace exists if the answer to the following question is affirmative: Does there exist a closed subspace of L^q which is not an $\mathscr{L}_q$ space?).

4. Linear topological investigation of the spaces $X_{p,w}$.

Let w be any fixed sequence of positive numbers, and let $2 < p < \infty$. We begin our investigations by showing that the "natural" block bases of $X_{p,w}$ span spaces isometric to $X_{p,w'}$ for some w', and are the ranges of norm-one orthogonal projections.

Let (g_n) be the unit-vectors in $X_{p,w}$ (i.e. $(g_n)_j = \delta_{nj}$ for all n,j positive integers). It is obvious that (g_n) is an unconditional (but incidentally not necessarily) a normalized basis of $X_{p,w}$. Now we define

$$\langle x,y\rangle = \sum x_i\bar{y}_i w_i^2 \text{ for all } x,y \in X_{p,w}$$

(where $x = (x_i) = \sum x_i g_i$ and similarly for y; $\bar{y}_i$ denotes the complex conjugate of y_i). Thus the norm on $X_{p,w}$ is the maximum of two norms, $\|\cdot\|_2$ and $\|\cdot\|_p$, where

$$\|x\|_p = (\sum |x_i|^p)^{1/p} \quad \text{and} \quad \|x\|_2 = \langle x,x\rangle^{1/2} = (\sum |x_i|^2 w_i^2)^{1/2}.$$

By the "natural" block bases of $X_{p,w}$, we refer to sequences consisting of the sums of disjoint blocks of the elements $w_n^{2/(p-2)} g_n$. (The choice of the constants $w_n^{2/(p-2)}$ comes from setting $c_n^p = c_n^2 w_n^2$.)

Our next result is a crucial tool in determining the linear topological properties of the spaces $X_{p,w}$.

LEMMA 7. *Let $E_1, E_2, \cdots$ be a sequence of disjoint finite subsets of the integers. For each j, put*

$$f_j = \sum_{n\in E_j} w_n^{2/(p-2)} g_n, \quad \beta_j = (\sum_{n\in E_j} w_n^{2p/(p-2)})^{(p-2)/2p}, \quad \textit{and}$$

$\tilde{f}_j = \|f_j\|_p^{-1} f_j \; (= \beta_j^{-2/(p-2)} f_j)$. *Let Y denote the closed linear span of the $\tilde{f}_j$'s in $X_{p,w}$. Then $(\tilde{f}_j)$ is an unconditional basis for Y, isometrically equivalent to the unit-vectors-basis of $X_{p,(\beta_j)}$ and there is a projection P from $X_{p,w}$ onto Y with $\|P\| = 1$.*

PROOF. Put $\delta_j = \beta_j^{2p/p-2}$ for all j. Note that

$$\delta_j = \sum_{n \in E_j} w_n^{2p/(p-2)} = \| f_j \|_p^p = \| f_j \|_2^2.$$

Now let $\lambda_1, \cdots, \lambda_n$ be given scalars. Then

$$\| \sum \lambda_j f_j \|_p^p = \| \sum_j \sum_{n \in E_j} \lambda_j w_n^{2/(p-2)} g_n \|_p^p$$

$$= \sum_j |\lambda_j|^p \sum_{n \in E_j} w_n^{2p/(p-2)} = \sum |\lambda_j|^p \delta_j, \qquad \text{and}$$

$$\| \sum \lambda_j f_j \|_2^2 = \sum |\lambda_j|^2 \sum_{n \in E_j} w_n^{4/(p-2)} w_n^2 = \sum \lambda_j^2 \delta_j.$$

Hence

$$\| \sum \lambda_j f_j \| = \max\{ (\sum |\lambda_j|^p \delta_j)^{1/p}, \quad (\sum |\lambda_j|^2 \delta_j)^{1/2} \}, \tag{5}$$

from which it follows immediately that $(\tilde{f}_j)$ is isometrically equivalent to the unit-vectors-basis of $X_{p,(\beta_j)}$.

We shall now show that orthogonal projection yields a projection of norm one from $X_{p,w}$ onto Y.

We define $P: X_{p\,w} \to Y$ by

$$Px = \sum_j < x, f_j > \| f_j \|_2^{-2} f_j \quad \text{for all } x \in X_{p,w}. \tag{6}$$

To show that P is a well-defined projection onto Y, of norm-one, it suffices to show that fixing $x \in X_{p,w}$, the series in (6) converges in both the norms $\| \cdot \|_2$ and $\| \cdot \|_p$, and that $\| Px \|_r \leqq \| x \|_r$ for $r = 2$ and $r = p$. But since P is orthogonal projection with respect to $\langle \cdot \rangle$, we have immediately that the series converges in $\| \cdot \|_2$ and that $\| Px \|_2 \leqq \| x \|_2$.

Now fix j, and put

$$\lambda_j = \langle x, f_j \rangle \| f_j \|_2^{-2} = \frac{1}{\delta_j} \sum_{n \in E_j} x_n w_n^{2(p-1)/(p-2)}$$

(where $x = (x_n)$). Then

$$|\lambda_j|^p = \delta_j^{-p} \; | \sum_{n \in E_j} x_n w_n^{2(p-1)/(p-2)} |^p$$

$$\leqq \delta_j^{-p} \sum_{n \in E_j} |x_n|^p \, (\sum_{n \in E_j} w_n^{2q(p-1)/(p-2)})^{p/q}$$

by Hölder's inequality. But

$$\delta_j^{-p} (\sum_{n \in E_j} w_n^{2q(p-1)/(p-2)})^{p/q} = \delta_j^{p/q-p} = \delta_j^{-1}.$$

Hence

$$|\lambda_j|^p \leqq \delta_j^{-1} \sum_{n \in E_j} |x_n|^p. \tag{7}$$

Thus by (5) and (7)

$$\begin{aligned}\|Px\|_p^p &= \|\sum \lambda_j f_j\|_p^p \\ &= \sum|\lambda_j|^p\delta_j \leqq \sum_j \sum_{n\in E_j}|x_n|^p \\ &\leqq \|x\|_p^p.\end{aligned}$$

Q.E.D.

REMARK. It follows from Lemma 7 that if $\sum w_n^{2p/(p-2)} = \infty$, then there exists a subspace Y of $X_{p,w}$ isometric to l^2 and a projection of norm one from $X_{p,w}$ onto Y. We simply choose $E_1, E_2, \cdots$ such that $1 \leqq (\sum_{m\in E_j} w_m^{2p/(p-2)})^{(p-2)/2p}$ and let $Y = [\tilde{f}_j]$ where the f_j's are defined as above.

COROLLARY 8. *Let $p>2$ and let w satisfy (1). Then for all positive integers n, there exists a basic sequence (h_j) in $X_{p,w}^*$ (resp. $(\tilde{f}_j)$ in $X_{p,w}$) equivalent to the usual basis of l^2, such that for any n distinct elements $h_{i_1},\cdots,h_{i_n}$ (resp. $f_{i_1},\cdots,f_{i_n}$), $(h_{i_1},\cdots,h_{i_n})$ is isometrically equivalent to the unit-vectors-basis of l_n^q (resp. $(\tilde{f}_{i_1},\cdots,\tilde{f}_{i_n})$ is isometrically equivalent to the unit-vectors basis of l_n^p).*

PROOF. Fix n. Since w satisfies (1), we may choose an infinite sequence $E_1, E_2, \cdots$ of disjoint finite subsets of the integers, such that for all j, putting

$$\beta_j = (\sum_{m\in E_j} w_m^{2p/(p-2)})^{(p-2)/2p},$$

then

$$2^{-1}n^{(2-p)/2p} \leqq \beta_j \leqq n^{(2-p)/2p}. \tag{8}$$

Now put $\tilde{f}_j = \beta_j^{-2/(p-2)}\sum_{m\in E_j} w_m^{2/(p-2)} g_m$ for all j. Then by Lemma 7, there exists a projection P from $X_{p,w}$ onto Y, the closed linear span of the $\tilde{f}_j$'s, of norm 1, and $(\tilde{f}_j)$ is isometrically equivalent to the unit-vectors-basis of $X_{p,(\beta_j)}$. Since $2^{-1}n^{(2-p)/2p} \leqq \beta_j \leqq 1$, the unit-vectors-basis of $X_{p,(\beta_j)}$ is equivalent to the usual basis of l^2, and each unit vector has norm one.

Now let $P_j: Y \to Y$ be the projection with one-dimensional range defined by $P_j(\sum x_i\tilde{f}_i) = x_j\tilde{f}_j$, and put $h_j = (P_jP)^*(\tilde{f}_j)$ for all j. Then since P and P_j have norm one for all j, (h_j) is equivalent to the usual basis for l^2.

Now let n distinct positive integers $i_1 \cdots i_n$ be given. Then for any n scalars $x_1, \cdots, x_n$;

$$\begin{aligned}\left(\sum_{j=1}^n |x_j|^2\beta_{i_j}^2\right)^{1/2} &\leqq \left(\sum_{j=1}^n |x_j|^p\right)^{1/p}\left(\sum_{j=1}^n \beta_{i_j}^{2p/(p-2)}\right)^{(p-2)/2p} \\ &\leqq \left(\sum_{j=1}^n |x_j|^p\right)^{1/p},\end{aligned}$$

where the first inequality follows from Hölder's inequality and the second one from (8). Hence $\| \sum_{j=1}^{n} x_j \tilde{f}_{i_j} \| = (\sum_{j=1}^{n} |x_j|^p)^{1/p}$, so $(\tilde{f}_{i_1} \cdots, \tilde{f}_{i_n})$ is isometrically equivalent to the usual basis of l_n^p. Since the unit-vectors of $X_{p,(\beta_j)}$ have norm one, $P_{i_1} + \cdots + P_{i_n}$ is a projection of norm one. Hence $(h_{i_1}, \cdots, h_{i_n})$ is isometrically equivalent to the dual basis of $(\tilde{f}_{i_1}, \cdots, \tilde{f}_{i_n})$, i.e. to the usual basis of l_n^q. Q.E.D.

Let us say that a Banach space X satisfies P_2 if given any basic sequence (e_n) in X equivalent to the usual l^2 basis, then for all $\varepsilon > 0$, there exists a subsequence (e_{n_i}) with (e_{n_i}) $(1 + \varepsilon)$-equivalent to the usual l^2 basis.

Now let $p > 2$ and let $w = (w_n)$ be a sequence satisfying (1). It follows easily from Corollary 8 that $X_{p,w}^*$ is not isomorphic to any Banach space satisfying P_2.

THEOREM 9. *Let p and w be as above, and let A be a closed subspace of l^p. Then $X_{p,w}$ is not a continuous linear image of $(l^2 \oplus l^2 \oplus \cdots)_p \oplus A$.*

It follows from this result that $X_{p,w}$ is not a continuous linear image of $A \oplus l^2$. Thus in the language of [17], $X_{p,w}$ is a closed subspace of the direct sum of two totally incomparable Banach spaces, yet $X_{p,w}$ is not isomorphic to the direct sum of a subspace of each of the two spaces.

To prove Theorem 9, we prove the equivalent assertion that $X_{p,w}^*$ is not isomorphic to a subspace of $(l^2 \oplus l^2 \oplus \cdots)_q \oplus B$ where B is isometric to a quotient space of l^q. In view of our remark preceding Theorem 9, this is in turn an immediate consequence of the following lemma:

LEMMA 10. *Let $1 < q < 2$ and let B be isometric to a quotient space of l^q. Then $(l^2 \oplus l^2 \oplus \cdots)_q \oplus B$ satifies P_2.*

We break the proof into a number of steps, most of which involve standard "sliding hump" arguments.

SUBLEMMA 1. l^2 satisfies P_2. Moreover every sequence in l^2 tending to zero weakly, but not in norm, contains a subsequence which is a basic sequence equivalent to the usual basis for l^2.

This assertion follows from the arguments and results of Bessaga and Pelczynski [1].

SUBLEMMA 2. Let N be a fixed positive integer, and let $(l^2 \oplus \cdots \oplus l^2)_q$ be the direct sum of N-copies of l^2, in the l^q norm. Then $(l^2 \oplus \cdots \oplus l^2)_q$ satisfies P_2.

PROOF. The members of $(l^2 \oplus \cdots \oplus l^2)_q$ consist of N-tuples $x = (x_1, \cdots, x_N)$ where $x_i \in l^2$ for all i. Let $P_i(x) = x_i$ for all such x. Now let (e_n) be a basic sequence equivalent to the usual basis for l^2. Let (e'_n) be a subsequence of (e_n) so that for each i, $\lim_{n\to\infty}\|P_i(e'_n)\|$ exists. Put $s_i = \lim_{n\to\infty}\|P_i(e'_n)\|$ for $1 \leqq i \leqq N$, and $s = \lim_{n\to\infty}\|e'_n\|$. Then of course $s = (\sum_{i=1}^N s_i^q)^{1/q} > 0$. Now let $\varepsilon > 0$. Choose ε_1 such that $(1+2\varepsilon_1)/(1-\varepsilon_1) < 1+\varepsilon$. For each i, $P_i(e'_j) \to 0$ weakly as $j \to \infty$. Thus in virtue of Sublemma 1, we may choose a subsequence (e''_n) of (e'_n) such that for all scalars $c_1, c_2, \cdots$ and all i, if $s_i \neq 0$, then

$$s_i(1-\varepsilon_1)(\Sigma|c_j|^2)^{1/2} \leqq \|\Sigma_j c_j P_i(e''_j)\| \leqq s_i(1+\varepsilon_1)(\Sigma|c_j|^2)^{1/2}$$

while if $s_i = 0$, then

$$\|\Sigma c_j P_i(e''_j)\| \leqq \frac{\varepsilon_1}{N}\cdot s(\Sigma|c_j|^2)^{1/2}.$$

Thus

$$\begin{aligned}\|\Sigma c_j e''_j\| &= \left(\sum_{i=1}^N \|\Sigma c_j P_i(e''_j)\|^q\right)^{1/q}\\ &\geqq s(1-\varepsilon_1)(\Sigma|c_j|^2)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\|\Sigma c_j e''_j\| &\leqq (1+\varepsilon_1)(\Sigma|s_i|^q)^{1/q}(\Sigma|c_j|^2)^{1/2} + \varepsilon_1 s(\Sigma|c_j|^2)^{1/2}\\ &= s(1+2\varepsilon_1)(\Sigma|c_j|^2)^{1/2}\end{aligned}$$

from which it follows that $(s^{-1}e''_j)$, and hence (e''_j), is $(1+\varepsilon)$-equivalent to the usual basis for l^2.

For the next sublemma, we need the following definition: Given any sequence $x = (x_j)$ with $x(j) \in l^2$ for all j, and any n, let $T_n(x) = y$ where $y = (y(j))$, $y(j) = x(j)$ for all $j \geqq n$ and $y(j) = 0$ for $j < n$.

SUBLEMMA 3. *Let $1 \leqq q < 2$ and let X be a subspace of $(l^2 \oplus l^2 \oplus \cdots)_q$, isomorphic to l^2. Then $\|T_n|X\| \to 0$ as $n \to \infty$.*

PROOF. Suppose the conclusion is false. Then since $\|T_n(x)\| \geqq \|T_{n+1}(x)\|$ for all $x \in (l^2 \oplus l^2 \oplus \cdots)_q$, we can choose a $\delta > 0$ and a sequence (x_n) of elements of X with $\|T_n x_n\| \geqq \delta$ and $\|x_n\| = 1$ for all n. Since X is reflexive, we can choose a weakly convergent subsequence $(x_n^{(1)})$ of (x_n). Since $\|T_n(x)\| \to 0$ for all $x \in (l^2 \oplus l^2 \oplus \cdots)_q$, we can choose $q_1 < q_2 < \cdots$ integers such that

$$\|T_{q_i}(x_{q_j}^{(1)})\| < \delta 2^{-(i+1)}$$

for all $i > j$. Now put $n_i = q_{2i}$ and $e_i = x_{q_{2i}} - x_{q_{2i-1}}$ for all $i = 1,2,\cdots$. We then have that $\|e_i\| \leqq 2$ for all i, $e_i \to 0$ weakly, and

(9) $\| T_{n_i}(e_j) \| < \delta 2^{-(i+1)}$ and $\| T_{n_j}(e_j) \| > \frac{3\delta}{4}$ for all i and j with $i > j$.

Now put $\tilde{e}_j = (T_{n_j} - T_{n_{j+1}})(e_j)$ for all j. Fot each j we have that $\| \tilde{e}_j \| \geqq \delta/2$ by (9). Hence we may choose $f_j \in (T_{n_j} - T_{n_{j+1}})^*[(l^2 \oplus l^2 \oplus \cdots)_q]^*$ such that $\| f_j \| \leqq 2/\delta$, and $f_j(\tilde{e}_j) = 1$. Of course $f_i(x) = f_i T_i(x)$ for all x. Thus $f_j(e_j) = f_j(\tilde{e}_j) = 1$. Moreover if $i > j$, then $|f_i(e_j)| \leqq 2/\delta \| T_i(e_j) \|$, hence by (9), $|f_i(e_j)| \leqq 1/2^i$.

Since $e_j \to 0$ weakly, we can choose $m_1 < m_2 < \cdots$ such that $|f_{m_i}(e_{m_j})| < 1/2^i$ for all $i < j$. Then choosing N such that $1/2^{N-4} \leqq \delta$, we have that

$$\sum_{\substack{i \geq N \\ i \neq j}} |f_{m_i}(e_{m_j})| < \frac{1}{2^{N-1}} \text{ holds for all } j \geqq N. \tag{10}$$

Now define a projection P onto $[\tilde{e}_{m_j} : j \geqq N]$ by $P(x) = \sum_{i=N}^{\infty} f_{m_i}(x) \tilde{e}_{m_i}$ for all $x \in (l^2 \oplus l^2 \oplus \cdots)_q$. Then by (10), $\| P(e_{m_i}) \| \geqq \delta/4$ for all $i \geqq N$, yet $(\tilde{e}_{m_j} : j \geqq N)$ is a basic sequence equivalent to the usual basis of l^q. Since every operator from l^2 to l^q is compact (c.f. page 206 of [16]), $P|X$ is compact, and thus $\| P(e_{m_i}) \| \to 0$ since (e_{m_i}) is a sequence of elements of X tending weakly to zero. This contradiction completes the proof of Sublemma 3. (The above argument holds in more general situations; see Remark 3 below.)

Lemma 10 now follows easily from the last two Sublemmas together with the observation that if $T : l^2 \to B$ is a given operator, then T is compact. (Indeed then T^* is an operator from a subspace of l^p into l^2, so T^* is compact by Theorem A2, page 206 of [16]).

REMARKS.

1. Given any $1 \leqq r < \infty$, we can of course define the property P_r by simply replacing "2" by "r" throughout the definition of P_2. Then the proof of Sublemma 2 shows that for any $1 \leqq q < \infty$ and any positive integer N, the N-fold sum of l^r, $(l^r \oplus \cdots \oplus l^r)_q$, satisfies P_r.

2. Given any $1 \leqq r < \infty$, let us say that a Banach space X satisfies Q_r if there exists a $K < \infty$ such that for every basic sequence (x_n) in B equivalent to the unit-vectors-basis (e_n) of l^r, there exists a subsequence (x_{n_j}) K-equivalent to (e_n). Evidently if X is isomorphic to a space satisfying P_r, then X satisfies Q_r.

Now Corollary 9 shows that if $2 < q < \infty$, and if w satisfies (1), then $X_{p,w}$ does not satisfy Q_2, hence, neither $l^p \oplus l^2$ nor $(l^2 \oplus l^2 \oplus \cdots)_q$ satisfies Q_2. Thus

Sublemma 3 is false for $2 < q < \infty$. For Sublemma 2 is true for all such q, and these two sublemmas constituted the proof that $(l^2 \oplus l^2 \oplus \cdots)_q$ satisfies P_2 for $1 < q < 2$.

3. Sublemma 3 holds for $q = 1$ also, of course; and the proof given shows that the following generalization holds:

Let $1 \leqq q < \infty$, *let* $X_1, X_2, \cdots$ *be given Banach spaces, and let* X *be a reflexive subspace of* $(\sum \oplus X_i)_q$ *such that every operator from* X *into* l^q *is compact* (automatic if $q = 1$). *Then* $\| T_n | X \| \to 0$ *as* $n \to \infty$, (*where* $T_n(x)(j) = x(j)$ *for all* $j \geqq n$, $T_n(x)(j) = 0$ *all* $j < n$; *all* n).

In particular, it follows that if $1 < q < 2$, $X_i \subset L^q$ for all i, and $X \subset (\Sigma \oplus X_i)_q$ is such that no subspace of X is isomorphic to l^q, then X is isomorphic to a subspace of $(X_1 \oplus \cdots \oplus X_n)_q$ for some n. (For then every operator from X to l^q is compact, c.f. page 211 of [16]). Since Sublemma 2 is valid for all r, we also obtain the following result:

Let $1 \leqq q < r < \infty$. *Then* $(l^r \oplus l^r \oplus \cdots)_q$ *satisfies* P_r.

We now consider the intrinsic linear topological properties of the spaces $X_{p,w}$ for w saiisfying (1′). We wish to demonstrate that any two such spaces are isomorphic. Toward this end, we shall show that $X_{p,w}$ is isomorphic to its own square; in fact $X_{p,w}$ is isomorphic to a symmetric sum of itself.

DEFINITION. *Let* Y *be a given Banach space. The Banach space* $(Z, \| \cdot \|)$ *is said to be a symmetric sum of* Y *if* Z *is a subspace of* Y^∞ (*the space of all infinite sequences of elements of* Y) *satisfying the following properties*:

(i) $Y_0^\infty \subset Z$ *and* Y_0^∞ *is dense in* Z, *where* Y_0^∞ *consists of all members of* Y^∞ *that are ultimately zero.*

(ii) *For all* $(y_n) \in Z$, *permutations* σ *of the positive integers, and sequences* (ε_n) *of scalars with* $|\varepsilon_n| \leqq 1$ *for all* n, $(\varepsilon_n y_{\sigma(n)}) \in Z$ *and*

$$\| (\varepsilon_n y_{\sigma(n)}) \| \leqq \| (y_n) \| . \tag{11}$$

Given a norm $\| \cdot \| = s$ defined on Y_0^∞ and satisfying (11) for all $(y_n) \in Y_0^\infty$, there exists a unique symmetric sum of Y, call it Z, such that the norm on Z agrees with s on Y_0^∞. Accordingly, we refer to a symmetric sum (Z,s) of Y by $(Y \oplus Y \oplus \cdots)_s$, and refer to the norm s as a symmetric norm on Z. (For a description of norms on subspaces of Y^∞ equivalent to symmetric ones, see the first remark following the next result.)

PROPOSITION 11. *Let* $Z = (Y \oplus Y \oplus \cdots)_s$ *be a symmetric sum of the Banach* *space Y. Then $Z \sim Z \oplus Y \sim Z \oplus Z$. If Z is isomorphic to a complemented subspace of Y, then Z and Y are isomorphic.*

PROOF. The assertions are straightforward, except possibly the last one. We recall the result of Pelczynski [13]: *if each of two Banach spaces is isomorphic to its own square and a complemented subspace of the other, they are isomorphic.* Now obviously Y is isomorphic to a complemented subspace of Z, since in fact $Z \sim Z \oplus Y$. If Z is isomorphic to a complemented subspace of Y, then there exists a closed subspace A of Y such that $Y \sim Z \oplus A$. Hence $Y \oplus Y \sim Y \oplus Z \oplus A \sim Z \oplus A \sim Y$, so by this result of Pelczynski, $Y \sim Z$.

REMARKS.

1. If Y is the one-dimensional space, then a symmetric sum of Y, $(Y \oplus Y \oplus \cdots)_s$, is simply a canonical representation of a symmetric space as defined in [15]. Moreover, one has the following Proposition generalizing the known result for symmetric spaces (c.f. [19]): *Let ρ be a complete norm on a subspace Z of Y^∞ satisfying* (i) *and* (ii)′: *For all $(y_n) \in Z$ and permutations σ of the positive integers, $(y_{\sigma(n)}) \in Z$. Then there exists a symmetric norm s on Z equivalent to ρ.*

2. Proposition 5 shows that there exists a two-dimensional Banach space Y, a symmetric sum of Y, $(Y \oplus Y \oplus \cdots)_s$, and one-dimensional subspaces B_n of Y such that $(B_1 \oplus B_2 \oplus \cdots)_s$ is uncomplemented in Y. Indeed we let Y be l_2^∞ and let $(Y \oplus Y \oplus \cdots)_s$ be all sequences $((x_n, y_n)) \in Y^\infty$ such that $s((s_n, y_n)) < \infty$, where $s((x_n, y_n)) = \max\{(\Sigma |x_n|^p)^{1/p}, (\Sigma |y_n|^2)^{1/2}\}$. Now let $2 < p < \infty$, let w satisfy (1), and let $B_j = \{(x, w_j x)$: x is an arbitrary scalar$\}$. Then $(Y \oplus Y \oplus \cdots)_s$ is canonically isometric to $l^p \oplus l^2$ and $(B_1 \oplus B_2 \oplus \cdots)_s$ is canonically isometric to its subspace $\tilde{X}_{p,w}$.

Now of course if Y is a given Banach space, and $1 \leqq p < \infty$, then $(Y \oplus Y \oplus \cdots)_p$ is a symmetric sum of Y. However we are interested in a different sort of symmetric sum of the spaces $X_{p,w}$. *In the following, let $2 < p < \infty$ be fixed, and let $w = (w_n)$ be fixed satisfying* (1′). Recall that we defined $\|x\|_2 = (\Sigma |x_n|^2 w_n^2)^{1/2}$ and $\|x\|^p = (\Sigma |x_n|^p)^{1/p}$ for all $x = (x_n)$ in $X_{p,w}$.

DEFINITION. Let Z be the set of all members (y_n) of $X_{p,w}^\infty$ satisfying $\Sigma \|y_n\|_2^2 < \infty$ and $\Sigma \|y_n\|_p^p < \infty$, and define $s = \|\cdot\|$ on Z by

$$\|(y_n)\| = \max\{(\Sigma \|y_n\|_2^2)^{1/2}, (\Sigma \|y_n\|_p^p)^{1/p}\}.$$

It is trivial to verify that s is indeed a symmetric norm on Z. (Of course s depends on p and w.).

PROPOSITION 12. *$X_{p,w}$ is isomorphic to $Z = (X_{p,w} \oplus X_{p,w} \oplus \cdots)_s$.*

PROOF. By Proposition 11, we need only show that Z is isomorphic to a complemented subspace B of $X_{p,w}$. Since w satisfies (1′), we may choose $N_1, N_2, \cdots$ disjoint infinite subsets of the positive integers such that for each k,

$$\sum_{n \in N_k} w_n^{2p/(p-2)} = \infty \quad \text{and} \quad \lim_{\substack{n \to \infty \\ n \in N_k}} w_n = 0.$$

Fixing k, we may choose $E_1^k, E_2^k, \cdots$ disjoint finite subsets of N_k such that putting $\beta_{j,k} = (\sum_{n \in E_j^k} w_n^{2p/(p-2)})^{(p-2)/2p}$,
then

$$(12) \qquad w_j \leqq \beta_{j,k} \leqq 2w_j \quad \text{for all } j.$$

Now let $\tilde{f}_{j,k} = (\beta_{j,k})^{-2/(p-2)} \sum_{n \in E_j^k} w_n^{2/(p-2)} g_n$, let X_k be the closed linear span of $\{\tilde{f}_{j,k}: j = 1,2,\cdots\}$, and let B be the closed linear span of $\{\tilde{f}_{j,k}: j = 1,2,\cdots; k = 1,2,\cdots\}$.

By Lemma 7, B is complemented in $X_{p,w}$. It is also easily seen that $y \in B$ if and only if there exists a sequence (y_k) with $y_k \in X_k$ for all k with $\sum \| y_k \|_2^2 < \infty$ and $\sum \| y_k \|_p^p < \infty$, and $y = \sum_{k=1}^{\infty} y_k$; if $y \in B$, this sequence (y_k) is moreover unique, and

$$(13) \qquad \| y \| = \max \{ (\sum \| y_k \|_2^2)^{1/2}, \ (\sum \| y_k \|_p^p)^{1/p} \}.$$

Now again fixing k, by Lemma 7 $(\tilde{f}_{j,k})_{j=1}^{\infty}$ is isomorphically equivalent to the unit-vectors-basis of $X_{p,(\beta_{j,k})_{j=1}^{\infty}}$. But by (12), the unit-vectors-basis of $X_{p,(\beta_{j,k})_{j=1}^{\infty}}$ is 2-equivalent to the unit-vectors-basis of $X_{p,w}$. It follows from (12) that defining $T_k: X_k \to X_{p,w}$ by $T_k(y) = \sum_j \alpha_j g_j$ if $y = \sum_j \alpha_j \tilde{f}_{j,k}$, then T_k is an isomorphism with

$$(14) \qquad \frac{\| y \|}{2} \leqq \| T_k(y) \| \leqq \| y \| \quad \text{for all } y \in X_k.$$

We may now define an isomorphism $T: B \to Z$ as follows: for each $b \in B$, choose the unique sequence (y_k) with $y_k \in X_k$ for all k and $b = \sum y_k$. Let $[T(b)]_k = T_k(y_k)$ for all k. It then follows easily from (13) and (14) that T is a surjective linear map with $\frac{\| b \|}{2} \leqq \| T(b) \| \leqq \| b \|$ for all $b \in B$. Q.E.D.

THEOREM 13. *Let $2 < p < \infty$, and let w and w' satisfy (1'). Then $X_{p,w}$ is isomorphic to $X_{p,w'}$.*

PROOF. By Propositions 11 and 12, $X_{p,w}$ and $X_{p,w'}$ are each isomorphic to their own square. Hence by the result of Pelczynski mentioned in the proof of Proposition 11, we need only show that each is isomorphic to a complemented subspace of the other. It suffices by symmetry, to show that $X_{p,w'}$ is isomorphic to a complemented subspace of $X_{p,w}$. Choose $E_1, E_2, \cdots$ disjoint finite subsets of the integers such that putting

$$\beta_j = \Big(\sum_{n \in E_j} w_n^{2p/(p-2)} \Big)^{(p-2)/2p}$$

for all j, then $w'_j \leqq \beta_j \leqq 2w'_j$ for all j. Then defining f_j as in Lemma 7, it follows immediately from that result, that the closed linear span of the f_j's is complemented and isomorphic to $X_{p,w'}$. Q.E.D.

REMARK. We actually have that there exists an absolute constant K such that $d(X_{p,w}, X_{p,w'}) \leqq K$ for any $2 < p < \infty$ and any w, w' satisfying (1').

We wish finally to consider the new $\mathscr{L}_p$ spaces generated by our methods. Let $p > 2$, and let X_p denote $X_{p,w}$ for any sequence $w = (w_n)$ satisfying (1'). We then define X_q by $X_q = X_p^*$. Our results of course show that X_p is different isomorphically from the previously known $\mathscr{L}_p$ spaces. We obtain another new isomorphism type among the $\mathscr{L}_p$ spaces as follows: For each positive integer n, let $B_{p,n}$ consist of all square summable sequences (x_j) of scalars under the norm

$$\| (x_j) \|_{B_{p,n}} = \max \{ n^{-(p-2)/2p} (\sum | x_j |^2)^{1/2}, (\sum | x_j |^p)^{1/p} \}.$$

Now of course $B_{p,n}$ is isomorphic to l^2. However the span of any n-unit-vectors in $B_{p,n}$ is isometric to l_n^p (whence $d(B_{p,n}, l_n^2) \to \infty$ as $n \to \infty$). Since $B_{p,n}$ is none other than $X_{p,\beta}$ where $\beta(j) = n^{-(p-2)/2p}$ for all j, it follows from Lemma 7 and the proof of Corollary 8 that for each n, there exists a subspace $\tilde{B}_{p,n}$ of X_p with $d(\tilde{B}_{p,n}, B_{p,n}) \leqq 2$ and a projection of norm one from X_p onto $\tilde{B}_{p,n}$. Thus defining $B_p = (B_{p,1} \oplus B_{p,2} \oplus \cdots)_p$, B_p is isomorphic to a complemented subspace of L^p and evidently not isomorphic to l^2, so B_p is an $\mathscr{L}_p$ space by the results of [10]. For $1 < p < 2$, define $B_p = B_q^*$.

COROLLARY 14. *Let $1 < p < \infty$, $p \neq 2$. Then $L^p, l^p, l^p \oplus l^2$, $(l^2 \oplus l^2 \oplus \cdots)_p$, X_p, and B_p are mutually non-isomorphic.*

PROOF. It suffices to prove this for $2 < p < \infty$. (It is proved in [7] that the first four spaces listed are mutually non-isomorphic.) Now B_p^* is not isomorphic to any Banach space satisfying P_2 (where P_2 is defined preceding Theorem 9). Hence, by the proof of Theorem 9, B_p is not a continuous linear image of $(l^2 \oplus l^2 \oplus \cdots)_p \oplus A$ where A is any subspace of l^p. Thus Theorem 9 implies that neither B_p nor X_p is isomorphic to l^p, $l^p \oplus l^2$, or $(l^2 \oplus l^2 \oplus \cdots)_p$. By the proof of Sublemma 3 of Lemma 10 (c.f. the third remark following Lemma 10), every subspace of B_p^* is either isomorphic to l^2 or contains a subspace isomorphic to l^q. Since L^q contains a subspace isomorphic to l^r if $q < r < 2$ (c.f. [7]), $B_p \not\sim L_p$. Finally to see that $X_p \not\sim B_p$ and $X_p \not\sim L^p$, we observe that $(l^2 \oplus l^2 \oplus \cdots)_p$ is isomorphic to a complemented subspace of B_p. (This follows immediately from the remark following Lemma 7.) However $(l^2 \oplus l^2 \oplus \cdots)_p$ is not isomorphic to a subspace of X_p, in fact we have the

LEMMA. *$(l^2 \oplus l^2 \oplus \cdots)_p$ is not isomorphic to a subspace of $l^p \oplus l^2$.*

To see this, let $Y_n = \{y \in (l^2 \oplus l^2 \oplus \cdots)_p : y(j) = 0 \text{ for all } j < n\}$. Let P be the projection of $l^p \oplus l^2$ onto l^2 with kernel l^p. Let $T: (l^2 \oplus l^2 \oplus \cdots)_p \to l^p \oplus l^2$ be a given operator. Since every operator from l^p into l^2 is compact (c.f. p. 206 of [16]), it follows that $\| PT | Y_n \| \to 0$ as $n \to \infty$. Hence letting

$$Q = I - P, \ \| (T - QT) | Y_n \| \to 0$$

as $n \to \infty$. Thus if T were one-to-one with closed range, we would have that for n sufficiently large, Y_n would be isomorphic to a subspace of $QT | Y_n$, i.e., to l^2. But l^p is isomorphic to a subspace of Y_n, so this is impossible.

This completes the proof of the Lemma and hence of Corollary 14.

Now let $Z_p = (l^2 \oplus l^2 \oplus \cdots)_p$ and $Y_p = (X_p \oplus X_p \oplus \cdots)_p$. In addition to the two new spaces mentioned above, we also obtain the $\mathscr{L}_p$ spaces

$$Z_p \oplus X_p, \ B_p \oplus X_p, \text{ and } Y_p.$$

Again fix $2 < p < \infty$. J. Lindenstrauss and A. Pelczynski have recently proved that L^p is not isomorphic to a subspace of Z_p [8]. Since Y_p is isomorphic to a subspace of Z_p and the three spaces mentioned above are all isomorphic to complemented subspaces of Y_p, all of these spaces are non-isomorphic to L^p and consequently to the other previously known $\mathscr{L}_p$ spaces by Corollary 14. They are also non-isomorphic to X_p, since Z_p is a factor of all of them, yet Z_p is not isomorphic to a subspace of X_p.

We have recently proved that l^r is isomorphic to a subspace of X_q for all $q < r < 2$. (Details of this will appear elsewhere). It follows that X_p is not a continuous linear image of B_p, and consequently the above three spaces are all non-isomorphic to B_p. We do not know if these three spaces are also mutually non-isomorphic. More information on subspaces of X_q seems to be required. In this connection, the following characterization of X_q may be of some use: Let (w_n) be a sequence satisfying (1) (for fixed $2 < p < \infty$) and let (g_n^*) be the dual basis of $X_{p,w}$. (Thus $X_p \sim X_{p,w}^*$ where as always $1/p + 1/q = 1$.) Then it can be seen that for any sequence of scalars (c_n), $\Sigma c_n g_n^*$ converges if and only if

$$\sum_{n=1}^{\infty} \min\left\{\frac{|c_n|^2}{w_n^2}, |c_n|^q\right\} < \infty$$

if and only if

$$\sum_{n=1}^{\infty} \phi_n(|c_n|) < \infty,$$

where

$$\phi_n(x) = \min\left\{\frac{x^2}{w_n^2}, \frac{2}{q}x^q\right\}$$

for all $x \geqq 0$. (The ϕ_n's are thus convex functions.) Then if we define

$$|||\, \Sigma c_n g_n^* ||| = \inf\left\{\frac{1}{|\lambda|}: \sum_{n=1}^{\infty} \phi_n(|\lambda c_n|) \leqq 1\right\},$$

then $|||\cdot|||$ is equivalent to the dual norm on $(X_{p\,w})^*$.

APPENDIX.

Let p be fixed throughout with $2 < p < \infty$. We present here a constructive proof that $\tilde{X}_{p,w}$ is uncomplemented in $l^p \oplus l^2$ if w satisfies (1), and also obtain thereby another proof that there exists an uncomplemented subspace of l^p isomorphic to l^p.

LEMMA A1. *Let n be a positive integer and let (a_{ij}) be an $n \times n$ matrix of scalars, and let $K > 0$ be such that for all scalars $x_1, \cdots, x_n$,*

$$\Big(\sum_i \Big|\sum_j a_{ij}x_j\Big|^2\Big)^{1/2} \leqq K\Big(\sum |x_i|^p\Big)^{1/p}.$$

Then

$$\sum_i \Big(\sum_j |a_{ij}|^2\Big)^{p/(p-2)} \leqq K^{2p/(p-2)},$$

hence also

$$\Sigma\,(|a_{ii}|^{2p/(p-2)}) \leqq K^{2p/(p-2)}.$$

PROOF. Suppose first that (a_{ij}) is diagonal; i.e. $i \neq j \Rightarrow a_{ij} = 0$. Put $x_i = |a_{ii}|^{2/(p-2)}$ for all i. Then

$$(\Sigma |a_{ii}x_i|^2)^{1/2} = (\Sigma |a_{ii}|^{2p/(p-2)})^{1/2} \leqq K (\Sigma |a_{ii}|^{2p/(p-2)})^{1/p}$$

or

$$\Sigma (|a_{ii}|^{2p/(p-2)}) \leqq K^{2p/(p-2)} .$$

Now for the general case: fix $x \in l_n^p$ with $\|x\| \leqq 1$ and let $r_1, \cdots, r_n$ be the first n Rademacher functions. We use only the fact that $r_1, \cdots, r_n$ are orthonormal real-valued functions in $L^2[0,1]$ with $|r_i(t)| = 1$ for all i and t. Then for each t, $0 \leqq t \leqq 1$,

$$\sum_i | \sum_j a_{ij}x_jr_j(t)|^2 \leqq K^2,$$

hence

$$K^2 \geqq \sum_i \int_0^1 | \sum_j a_{ij}x_jr_j(t)|^2 dt$$

$$= \sum_i \sum_j |a_{ij}|^2 |x_j|^2 = \sum_j (\sum_i |a_{ij}|^2) |x_j|^2 .$$

Thus putting $a'_{jj} = (\Sigma_i |a_{ij}|^2)^{1/2}$ for all j, $a'_{ij} = 0$ for $i \neq j$, $(\Sigma |a'_{jj}x_j|^2)^{1/2} \leqq K \; \Sigma \; |x_j|^p)^{1/p}$ for any $x \in l_n^p$, whence since the Lemma has been proved for diagonal matrices,

$$\Sigma |a'_{jj}|^{2p/(p-2)} = \sum_j (\sum_i |a_{ij}|^2)^{p/(p-2)} \leqq K^{2p/(p-2)}.$$

Q.E.D.

REMARKS. Lemma A1 of course generalizes immediately to infinite matrices as well. We may reformulate its conclusion as follows:

For every operator $T: l^p \rightarrow l^2$, $\Sigma_n (\| Te_n \|^{2p/(p-2)}) \leqq \| T \|^{2p/(p-2)}$ *where* (e_n) *denotes the usual basis of* l^p.†

It follows immediately that if $T: l_n^p \rightarrow l_n^2$ is one-one with $\|T\| \leqq 1$, then $\|T^{-1}\| \geqq n^{(p-2)/2p}$, whence we obtain the known result that $d(l_n^p, l_n^2) = n^{(p-2)/2p}$.

For the next results we recall that $(e_1, \cdots, e_n)$ (resp. $(b_1, \cdots, b_n)$) denotes the usual basis of l_n^p (resp. l_n^2), and Z_n denotes $l_n^p \oplus l_n^2$, under the norm $\|x \oplus y\| = \max \{\|x\|, \|y\|\}$

LEMMA 2. *Let n be fixed, let* $w_1, \cdots, w_n$ *be positive numbers with* $w_i \leqq 1/2$ *for all* i, *and let* X_n *denote the span of* $\{e_i + w_ik_i : 1 \leqq i \leqq n\}$ *in* Z_n. *Then for all projections* $P: Z_n \rightarrow X_n$,

† This result has also been obtained independently by E. Rietz (unpublished).

$$2\|P\| \geqq \min. \{(\sup_{1\leqq i\leqq n} |w_i|)^{-1}, \left(\sum_{i=1}^{n} |w_i|^{2p/(p-2)}\right)^{(p-2)/2p}\}$$

PROOF. Let $F_1,\cdots,F_n \in X_n^*$ be defined by $F_i(e_j + w_jb_j) = \delta_{ij}$ for all i,j. Then $\|F_i\| = 1$ for all n. Define (a_{ij}) by $a_{ij} = w_iF_i(Pe_j)$. Since P is a projection

$$F_jP(e_j + w_jb_j) = 1 = F\ P(e_j) + w_jF_jP(b_j).$$

Since $|F_jP(b_j)| \leqq \|P\|$, we thus have that

$$|a_{jj}| = w_j|F_j(Pe_j)| \geqq w_j(1 - w_j\|P\|).$$

It is easily seen that the matrix (a_{ij}) satisfies the hypotheses of Lemma A1 for $K = \|P\|$. Hence if $2\|P\| \leqq (\sup_{1\leqq i\leqq n}|w_i|)^{-1}$, then we obtain by Lemma 1 and the above inequalities that

$$\begin{aligned}2^{-2p/(p-2)} \sum |w_j|^{2p/(p-2)} &\leqq \sum |w_j|^{2p/(p-2)}(1 - |w_j|\,\|P\|)^{2p/(p-2)}\\ &\leqq \sum |a_{jj}|^{2p/(p-2)} \leqq \|P\|^{2p/(p-2)}\end{aligned}$$

or $(\sum |w_j|^{2p/(p-2)})^{(p-2)/2p} \leqq 2\|P\|$. This proves Lemma A2.

The above lemma enables us to construct an uncomplemented subspace of l^p, isomorphic to l^p, as follows. Fix n for the moment, and put $w_j = n^{-(p-2)/4p}$ for all j, $1 \leqq j \leqq n$. Lemma 2 implies that if P is a projection from Z_n onto X_n, then $\|P\| \geqq \frac{1}{2}n^{(p-2)/4p}$ (provided $n^{-(p-2)/4p} < \frac{1}{2}$). Hence it follows immediately that $(\sum_{n=1}^{\infty} \oplus X_n)_p$ is uncomplemented in $(\sum_{n=1}^{\infty} \oplus Z_n)_p$; our proof of Theorem 6 shows that both of the latter spaces are isomorphic to l^p. More generally, we obtain the following result (also implied by our Proposition 5 above):

PROPOSITION A3. *Let (w_n) be an infinite sequence satisfying* (1). *Let $\rho_n = \inf\{\|P\| : \rho : Z_n \to X_n$ is a surjective projection$\}$, where X_n is as defined in Lemma 2, for all n. Then $\rho_n \to \infty$. Consequently $X_{p,w}$ is uncomplemented in $l^p \oplus l^2$ and $(\sum \oplus X_n)_p$ is uncomplemented in $(\sum_{n=1}^{\infty} \oplus Z_n)_p$.*

PROOF. Let M be a given positive real number. Choose N such that $|w_i| \leqq (2M)^{-1}$ for all $i \geqq N$. Now choose N_1 such that $(\sum_{i=N}^{N_1} |w_i|^{2p/(p-2)})^{(p-2)/2p} \geqq 2M$ and fix $n \geqq N_1$. Let Z'_n denote the span of $\{e_i, b_i: N \leqq i \leqq n\}$ and X'_n the span of $\{e_i + w_ib_i: N \leqq i \leqq n\}$. It follows from Lemma A2 that if $P: Z'_n \to X'_n$ is a projection, then $\|P\| \geqq M$. Since there is a projection of norm one from X_n onto X'_n, it follows immediately that $\rho_n \geqq M$; hence $\rho_n \to \infty$ as $n \to \infty$, proving Proposition A3. Q.E.D.

Remarks. Let n and the w_i's be as in Lemma 2. One can obtain the correct order of magnitude for the norm of a projection P from Z_n onto X_n as follows: assume that $w_1 \leqq w_2 \cdots \leqq w_n$. If there exists a j (necessarily unique) with $1 \leqq j \leqq n-1$ such that

$$\sum_{i=1}^{j} w_i^{2p/(p-2)} \leqq w^{-2p/(p-2)} \quad \text{and} \quad \sum_{i=1}^{j+1} w_i^{2p/(p-2)} > w_{j+1}^{-2p/(p-2)}$$

then Lemma 2 implies that $2\|P\| \geqq \max\{(\sum_{i=1}^{j} w_i^{2p/(p-2)})^{(p-2)/(2p)}, w_{j+1}^{-1}\}$. ($\|P\|$ must be at least as large as the norm of a projection from Z_j onto X_j, or from Z_{j+1} onto X_{j+1}). However, if we define P by

$$P\left(\sum_{i=1}^{n} \lambda_i e_i + \beta_i b_i\right) = \sum_{i=1}^{j} \lambda_i(e_i + w_i b_i) + \sum_{i=j+1}^{n} w_i^{-1}\beta_i(e_i + w_i b_i),$$

for all scalars $\lambda_1, \cdots, \lambda_n, \beta_1, \cdots, \beta_n$, then

$$(\Delta) \qquad \|P\| \leqq 2\max\left\{(\sum_{i=1}^{j} w_i^{2p/(p-2)})^{(p-2)/(2p)}, w_{j+1}^{-1}\right\}.$$

On the other hand, if there is no such j, then $2\|P\| \geqq (\sum_{i=1}^{n} w_i^{2p/(p-2)})^{(p-2)/(2p)}$ for any projection P, but there is a projection of norm at most $(\sum_{i=1}^{n} w_i^{2p/(p-2)})^{(p-2)/(2p)} \leqq w_n^{-1} \leqq n^{(p-2)/(4p)}$.

In addition, the right side of the inequality (Δ) is less than or equal to $2n^{(p-2)/4p}$. Thus the example given preceding Proposition A3 gives the largest possible size of $\|P\|$ for a projection P from Z_n onto X_n, to within the constant $\frac{1}{4}$. Note that X_n is an n-dimensional subspace of the $2n$-dimensional space $Z_n = l_n^p \oplus l_n^2$. (The results of Sobczyk [20] show that in fact there exists a subspace K of l_n^p such that $\|P\| \geqq \frac{1}{2}(n^{(p-2)/2p} - 1)$ for any projection P from l_n^p onto K; however this space K seems to be difficult to write down explicitly. We suspect that this K is considerably different from the finite-dimensional spaces discussed in this paper.)

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